STA261 (Summer 2024) - Assignment 3

These problems are meant to test your understanding of the concepts in Module 3. They are *not* to be handed in. Some of these have been modified (or in some cases taken directly) from questions in the *Additional Resources* listed in the course syllabus, and no claims of originality are made.

---

1. Let $X_1, X_2 \overset{iid}{\sim} \text{Unif}(\theta, \theta+1)$ where $\theta \in \mathbb{R}$, and suppose we want to test $H_0 : \theta = 0$ versus $H_A : \theta > 0$. Consider two tests based on two rejection regions:

$$R_1 = \{(x_1, x_2) : x_1 > 0.95\}$$
$$R_2 = \{(x_1, x_2) : x_1 + x_2 > c\}$$

   Calculate the size of the first test, and find c so that both tests have the same size.

2. Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \text{Unif}(\theta, \theta+1)$ where $\theta \in \mathbb{R}$ and suppose we want to test $H_0 : \theta = 0$ versus $H_A : \theta > 0$ by rejecting H_0 when $X_{(n)} \geq 1$ or $X_{(1)} \geq c$, for some $c \in \mathbb{R}$. Find c so that this is a size-α test.

3. Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim}$ be a random sample from a Pareto(θ, ν) distribution, which has density

$$f_{\nu,\theta}(x) = \theta\nu^\theta \cdot x^{-\theta-1}, \quad x \geq \nu > 0, \quad \theta > 1.$$

   (a) Find the MLEs of ν and θ. You don't need the whole multivariate optimization business of Example 2.14. Instead, start by fixing θ and finding $\hat{\nu}$, and then maximize $L(\theta, \hat{\nu} \mid \mathbf{x})$ in θ. Then $(\hat{\theta}, \hat{\nu})$ will be your MLE.

   (b) Show that the LRT of $H_0 : \theta = 1$ versus $H_A : \theta \neq 1$ has a critical region of the form $\{\mathbf{x} : T(\mathbf{x}) \leq c_1 \,\text{or}\, T(\mathbf{x}) \geq c_2\}$ for some $0 < c_1, c_2 < \infty$, where

$$T(\mathbf{X}) = \log\left(\frac{\prod_{i=1}^n X_i}{(X_{(1)})^n}\right).$$

4. Suppose that X is in a location family with pdf $f_\mu(x) = f(x-\mu)$. Fix any $c \in \mathbb{R}$, and show that $\mu_1 \leq \mu_2$ implies $\mathbb{P}_{\mu_1}(X > c) \leq \mathbb{P}_{\mu_2}(X > c)$.

5. Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \mathcal{N}(\mu, \sigma^2)$ where σ^2 is unknown. Consider the two sided test $H_0 : \mu = \mu_0$ versus $H_A : \mu \neq \mu_0$.

   (a) Show that the test that rejects H_0 when $|\bar{X}_n - \mu_0| > t_{n-1,\alpha/2}\sqrt{S^2/n}$ is a size-α test.

   (b) Show that this test is an LRT. You know what the unrestricted MLE of (μ, σ^2) is from Module 2, and you can write down the restricted MLE of (μ, σ^2) without any calculations. The rest is just algebra.

6. Suppose $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \text{Bernoulli}(\theta)$ where $\theta \in (0,1)$. We want to test $H_0 : \theta \leq \frac{1}{2}$ versus $H_A : \theta > \frac{1}{2}$, rejecting H_0 when $\sum_{i=1}^n x_i \geq c$. Calculate the p-value if we observe 7 successes out of 10 trials, and decide whether we accept or reject H_0 at the 0.05 significance level.

7. Suppose $X \sim \text{Poisson}(\lambda)$ where $\lambda > 0$. We want to test $H_0 : \lambda \leq 1$ versus $H_A : \lambda > 1$, rejecting H_0 when $x \geq c$. Calculate the p-value if we observe $X = 3$, and decide whether we accept or reject H_0 at the 0.05 significance level.

8. Suppose $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \mathcal{N}(\mu, \sigma^2)$ where $\mu \in \mathbb{R}$ and $\sigma^2 > 0$. We want to test $H_0 : \mu = 261$ versus $H_A : \mu \neq 261$ using a two-sided t-test. Calculate the p-value if $n = 140$ and we observe $\bar{X}_{140} = 248$ and $S^2 = 20$, and decide whether we accept or reject H_0 at the 0.05 significance level.

9. Suppose $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \mathcal{N}(\mu, 1)$ where $\mu \in \mathbb{R}$. We want to test $H_0 : \mu = 261$ versus $H_A : \mu > 261$ using a one-sided Z-test. Calculate the p-value if $n = 140$ and we observe $\bar{X}_{140} = 262$, and decide whether we accept or reject H_0 at the 0.05 signifciance level.

10. Suppose $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \mathcal{N}(\mu, 1)$ where $\mu \in \mathbb{R}$. We want to test $H_0 : \mu = 261$ versus $H_A : \mu < 261$ at the 0.05 significance level using a one-sided Z-test. Determine the sample size n needed to obtain a Type II error of at most 0.1 if the true parameter is $\mu = 248$.

11. You might have heard of the AM-GM inequality, which says that for any $x_1, x_2, \ldots, x_n \geq 0$, the arithmetic mean always upper bounds the geometric mean:

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n}.$$

There are plenty of ways to prove this; it turns out that one of them uses LRTs.

    (a) Show that if any of the x_i's are zero, the inequality is trivially satisfied.

    (b) Suppose that $X_1, X_2, \ldots, X_n$ are independent with $X_i \sim \text{Exp}(\lambda_i)$. Calculate the LRT statistic $\lambda(\mathbf{X})$ of $H_0 : \lambda_1 = \lambda_2 = \cdots = \lambda_n$ versus H_A : the λ_i's aren't all equal.

    (c) Take any $\mathbf{x} \in \mathcal{X}^n$, argue that that $\lambda(\mathbf{x}) \leq 1$, and establish the AM-GM inequality.

12. Suppose $X \sim \text{Beta}(\theta, 1)$.

    (a) Suppose we want to test $H_0 : \theta \leq 1$ versus $H_A : \theta > 1$. Find the size of the test that rejects H_0 if $X > \frac{1}{2}$.

    (b) Find the UMP level-α test of $H_0 : \theta = 1$ versus $H_A : \theta = 2$.

    (c) Find the UMP level-α test of $H_0 : \theta \leq 1$ versus $H_A : \theta > 1$.

13. Prove Theorem 3.5.

14. Prove Theorem 3.8.

15. Suppose $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \mathcal{N}(\mu, \sigma^2)$ where $\mu \in \mathbb{R}$ and σ^2 is known, and we want to test $H_0 : \mu = \mu_0$ versus $H_A : \mu \neq \mu_0$ using a test that rejects H_0 when $|\bar{X} - \mu_0| / \sqrt{\sigma^2/n} > c$. How can we choose c and n to obtain a size 0.25 test with a maximum Type II error probability of 0.25 at $\mu = \mu_0 + \sigma$?

16. Suppose that the hypotheses of Theorem 3.1 hold, and that $T(\mathbf{X})$ has a continuous distribution. Show that when $H_0 : \theta = \theta_0$ is true, $p(\mathbf{X}) \sim \text{Unif}(0, 1)$, and interpret this fact.

17. Suppose $\mathcal{X} = \{1, 2, 3, 4\}$ and $\Theta = \{a, b\}$. Two mass functions on $\mathcal{X}$ – one for each value of $\theta \in \Theta$ – are specified in the following table:

	$x = 1$	$x = 2$	$x = 3$	$x = 4$
$p_a(x)$	1/3	1/6	1/12	5/12
$p_b(x)$	1/2	1/4	1/6	1/12

Suppose we observe $X \sim p_\theta$. Determine a UMP level-0.10 test for testing $H_0 : \theta = a$ versus $H_A : \theta = b$.

18. Supose that $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \Gamma(\alpha_0, \beta)$ where α_0 is known and $\beta > 0$. Determine a UMP level-α test for testing $H_0 : \beta = \beta_0$ versus $H_A : \beta = \beta_1$, assuming $\beta_1 > \beta_0$.

19. Recall the simple *linear regression* setup from Assignment 2 Q8, where

$$Y_i = \alpha + \beta x_i + \epsilon_i, \quad i = 1, \ldots, n,$$

where $\epsilon_1, \epsilon_2, \ldots, \epsilon_n \overset{iid}{\sim} \mathcal{N}\left(0, \sigma^2\right)$. We previously derived the MLEs of α and β. Every software implementation of linear regression will calculate the MLEs using those same formulas, and they'll also output a number of test statistics and p-values. We'll derive some of those here. It'll help to define $S_{xx} = \sum_{i=1}^n (\bar{x} - x_i)^2$.

(a) Using the formulation from Assignment 2 as inspiration, explain what the hypothesis $\beta = 0$ would correspond to in real life. In any scientific study that uses linear regression, why is this more appropriate as a null hypothesis than an alternative?

(b) Observe that $\hat{\beta}(\mathbf{Y}) = \sum_{i=1}^n d_i Y_i$, where $d_i = \frac{x_i - \bar{x}}{S_{xx}}$, and use that to show

$$\hat{\beta} \sim \mathcal{N}\left(\beta, \frac{\sigma^2}{S_{xx}}\right).$$

(c) Observe that $\hat{\alpha}(\mathbf{Y}) = \sum_{i=1}^n c_i Y_i$, where $c_i = \frac{1}{n} - \frac{(x_i - \bar{x})\bar{x}}{S_{xx}}$, and use that to show

$$\hat{\alpha} \sim \mathcal{N}\left(\alpha, \frac{\sigma^2}{n \cdot S_{xx}} \sum_{i=1}^n x_i^2\right).$$

(d) Define the i'th **residual from the regression** to be $\hat{\epsilon}_i := Y_i - \hat{\alpha}(\mathbf{Y}) - \hat{\beta}(\mathbf{Y}) x_i$, for $i = 1, \ldots, n$. Interpret this quantity and show that $\mathbb{E}\left[\hat{\epsilon}_i\right] = 0$. With a lot of algebra, one can also show that

$$\operatorname{Var}\left(\hat{\epsilon}_i\right) = \left(\frac{n-2}{n} + \frac{1}{S_{xx}}\left(\frac{1}{n}\sum_{j=1}^n x_j^2 + x_i^2 - 2(x_i - \bar{x})^2 - 2x_i\bar{x}\right)\right)\sigma^2.$$

(e) Define the **residual sum of squares (RSS)** as $\mathrm{RSS} = \sum_{i=1}^n \hat{\epsilon}_i$, and let $\hat{\sigma}^2 = \frac{1}{n}\mathrm{RSS}$. Show that $\mathbb{E}\left[\hat{\sigma}^2\right] = \frac{n-2}{n}\sigma^2$.

(f) Show that $\operatorname{Cov}\left(\hat{\alpha}, \hat{\epsilon}_i\right) = 0$ and $\operatorname{Cov}\left(\hat{\beta}, \hat{\epsilon}_i\right) = 0$. To save a lot of work, write

$$\hat{\epsilon}_i = \sum_{j=1}^n [\delta_{ij} - (c_j + d_j x_i)] Y_i,$$

where $\delta_{ij} = \mathbb{1}_{i=j}$. You can use the following fact without proof: if $Y_1, Y_2, \ldots, Y_n$ are uncorrelated random variables (not necessarily independent or Normally distributed) with $\operatorname{Var}\left(Y_i\right) = \sigma^2$ for all i, then $\operatorname{Cov}\left(\sum_{i=1}^n a_i Y_i, \sum_{i=1}^n b_i Y_i\right) = \left(\sum_{i=1}^n a_i b_i\right)\sigma^2$ for any constant a_i's and b_j's.

(g) Define $\tilde{S}^2 = \frac{1}{n-2}\sum_{i=1}^n \hat{\epsilon}_i$, and show it's an unbiased estimator of σ^2. This is like a weird version of the usual sample variance S^2. Similarly to S^2, one can show that $\frac{n-2}{\sigma^2}\tilde{S}^2 \sim \chi^2_{(n-2)}$.

(h) Explain why, in this particular case, it must be that the $\hat{\epsilon}_i$'s are independent of both $\hat{\beta}$ and $\hat{\alpha}$. Of course, it follows that $\tilde{S}^2$ itself is also independent of both $\hat{\beta}$ and $\hat{\alpha}$.

(i) Finally, show that

$$\frac{\hat{\alpha} - \alpha}{\sqrt{\tilde{S}^2\left(\sum_{i=1}^n x_i^2\right)/(n S_{xx})}} \sim t_{n-2}$$

and

$$\frac{\hat{\beta} - \beta}{\sqrt{\tilde{S}^2 / S_{xx}}} \sim t_{n-2}.$$