UNIVERSITY OF TORONTO
Faculty of Arts and Science

STA261H1: Probability and Statistics II
Midterm 2
July 30, 2024

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- Do not open this test until you are told to begin.
- Midterm 2 is closed-book; no aids are allowed.
- Whatever you write on the backs of these pages will not be graded, so you can use them for scratch work.
- There are six questions (worth a total of 30 points) on the midterm, plus one bonus question (worth five additional points). Take a quick scan through the questions first and prioritize your time accordingly.
- Show all of your work for full marks, and ensure your notation is legible, correct, and consistent with that used in the course. Use $\vec{X}$ and $\vec{x}$ to denote vectors.
- If you need to use a result from lecture, either refer to it by its name (if it is a named theorem), or briefly describe it.

Good luck!

1. (5 points) Answer the following.

   (a) (1 point) State the definition of a **pivotal quantity**.

   (b) (1 point) State the **Neyman-Pearson lemma**.

   (c) (2 point) Describe what is meant by a **goodness of fit test**, and give an example of one.

   (d) (1 point) State what is meant by a **type II error**.

2. (5 points) Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \text{Unif}\,(-\theta, \theta)$, where $\theta \in \mathbb{R}$.[1] Suppose we want to test $H_0 : \theta = \theta_0$ versus $H_A : \theta \neq \theta_0$ using a test characterized by the rejection region $R = \{\mathbf{x} \in \mathcal{X}^n : x_{(n)} > c\}$ for some $c > 0$.

   (a) (2.5 points) Compute the power function of the test.

   (b) (2.5 points) Find the c that makes this a size-α test.

[1]Recall that when $a < b$, the general $\text{Unif}\,(a, b)$ distribution has cdf given by $F_{a,b}(x) = \frac{b-x}{b-a} \cdot \mathbb{1}_{x \in (a,b)} + \mathbb{1}_{x \geq b}$. Also, if $X_1, \ldots, X_n$ are iid and continuous with cdf F_X, then $X_{(n)}$ has cdf $F_{X_{(n)}}(x) = (F_X(x))^n$.

3. (5 points) Assume that $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \mathcal{N}(\mu, \sigma^2)$, where $\mu \geq 1$ and σ^2 is known.[2]

(a) (2.5 points) Find the value of $b \in \mathbb{R} \setminus \{0\}$ that makes $(\bar{X}_n/b, \infty)$ into a lower one-sided $(1-\alpha)$-confidence interval for μ.

(b) (2.5 points) Even though we don't know μ, explain how we might use a visual diagnostic to assess the assumption that the data is $\mathcal{N}(\mu, \sigma^2)$-distributed to begin with.

[2]Recall that the $\mathcal{N}(\mu, \sigma^2)$ distribution has pdf given by $f_\mu(x) = \exp\left(-(x-\mu)^2/2\sigma^2\right)/\sqrt{2\pi\sigma^2}$, where $x \in \mathbb{R}$.

4. (5 points) Answer each of the following questions by writing YES or NO (1 point), and justify your answer in *at most* three sentences (1.5 points).

(a) Suppose we calculate a 95%-confidence interval $(L(\mathbf{X}), U(\mathbf{X}))$ for an unknown parameter $\theta \in \Theta$, and we observe $L(\mathbf{X}) = -10$ and $U(\mathbf{X}) = 7$. Is it true that $\mathbb{P}_\theta(-10 \leq \theta \leq 7) = 0.95$?

(b) Let $T(\mathbf{X})$ be sufficient for θ and let $\lambda(\mathbf{X})$ and $\lambda^*(T(\mathbf{X}))$ be the LRT statistics based on $\mathbf{X}$ and $T(\mathbf{X})$, respectively. If a size-α LRT rejects H_0 when $\lambda(\mathbf{X}) \leq 0.261$, will a test that rejects H_0 when $\lambda^*(T(\mathbf{X})) \leq 0.261$ also be a size-α LRT?

5. (5 points) Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \text{Bernoulli}(\theta)$, where $\theta \in (0,1)$.[3] Suppose we want a level-α test of the hypothesis $H_0 : \theta \leq \theta_0$ versus $H_A : \theta > \theta_0$.

(a) (4 points) Using the Karlin-Rubin theorem, show that for some $c_0 > 0$, the test with rejection region $R = \{\mathbf{x} \in \mathcal{X}^n : \sum_{i=1}^n x_i > c_0\}$ is a UMP level-α test.

(b) (1 point) How might you find an appropriate c_0 for the test above?

[3]Recall that the Bernoulli (θ) distribution has pmf given by $f_\theta(x) = \theta^x(1-\theta)^{1-x}$ for $x \in \{0,1\}$. It might also help to recall that the Bin (n, θ) distribution has pmf given by $\binom{n}{x}\theta^x(1-\theta)^{n-x}$ for $x \in \{0, 1, \ldots, n\}$.

6. (5 points) Suppose $\boldsymbol{\theta} = (\theta_1, \dots, \theta_k) \in \mathbb{R}^k$ and that we can produce a $(1-\alpha_j)$-confidence interval $(L_j(\mathbf{X}), U_j(\mathbf{X}))$ for each θ_j. Prove that if we choose $\alpha_1 = \cdots = \alpha_k = \alpha/k$, then

$$C(\mathbf{X}) = (L_1(\mathbf{X}), U_1(\mathbf{X})) \times \cdots \times (L_k(\mathbf{X}), U_k(\mathbf{X}))$$

is a $(1-\alpha)$-confidence region for $\boldsymbol{\theta}$, in the sense that $\mathbb{P}_{\boldsymbol{\theta}}(\boldsymbol{\theta} \in C(\mathbf{X})) \geq 1 - \alpha$.

Hint: One of De Morgan's laws implies that for any sets $A_1, \dots, A_n$, we have $\cap_{i=1}^n A_i = (\cup_{i=1}^n A_i^c)^c$.

7. (BONUS: 5 points) Suppose that $X_1, \ldots, X_n \overset{iid}{\sim} F_\theta$ are continuous and that θ is the *median* of F_θ: that is, $\mathbb{P}_\theta(X_i \leq \theta) = 1/2$. Find the smallest sample size n such that $(X_{(1)}, X_{(n)})$ is a $(1-\alpha)$-confidence interval for θ.