University of Toronto
STA261 (Summer 2021) - Quiz 5
August 10, 2021

- Quiz 5 is open-notes and consists of two questions. There are 15 points available. Take a quick scan through the questions first and prioritize your time accordingly.
- Show all of your work for full marks, and ensure your notation is legible, correct, and consistent with that used in the course. Be sure to clearly distinguish between random variables and constants, and between vectors and scalars (you can write $\mathbf{X}_n$ as $\vec{X}_n$ and $\mathbf{x}_n$ as $\vec{x}_n$).
- Upload clear, legible photos/screenshots of your handwritten answers to the questions within the time window, one question at a time.
- You may refer to the lecture slides and assignments. If you need to use a result from them, either refer to it by its number or its name (if it is numbered/named), or describe the result.

Good luck!

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1. (10 points) Let $X_1, X_2, \ldots, X_n$ be a random sample from a continuous distribution with pdf

$$f_\theta(x) = \theta x^{-(\theta+1)}, \quad x > 1, \quad \theta > 0.$$

(a) (1 point) Show that the Fisher information for this model is $I_1(\theta) = 1/\theta^2$.

(b) (2 points) The MLE of θ is $\hat{\theta} = 1/\log\left(\left(\prod_{i=1}^n X_i\right)^{1/n}\right)$. Verify that sufficient regularity conditions hold for this MLE to be asymptotically efficient.

(c) (2 points) Find the asymptotic distribution of $\left(\prod_{i=1}^n X_i\right)^{1/n}$.

(d) (2.5 points) Derive the Wald statistic for testing $H_0 : \theta = \theta_0$ versus $H_A : \theta \neq \theta_0$, and state its asymptotic distribution under H_0. Simplify your answer as much as possible.

(e) (2.5 points) Derive an approximate $(1-\alpha)$-confidence interval based on your Wald statistic.

2. (5 points) Answer each of the following questions by writing YES or NO (1 point), and justify your answer in *at most* three sentences (1.5 points).

   (a) (2.5 points) Alice, a friend of yours, decides to carry out all three asymptotic size-α tests of the "trinity" for testing $H_0 : \theta = \theta_0$ versus $H_A : \theta \neq \theta_0$. To her dismay, she finds that one test tells her to reject H_0, while the other two tell her to fail to reject H_0. Has Alice necessarily done something wrong?

   (b) (2.5 points) Bob, another friend of yours, has studied the CLT and claims that for any model with $\mathbb{E}_\theta[X_i] = \theta$ and $\mathrm{Var}_\theta(X_i) = \sigma^2$, the sample mean $\bar{X}_n$ follows a $\mathcal{N}\left(\theta, \sigma^2/n\right)$ distribution when n is large enough. Is Bob correct?