University of Toronto
STA261 (Summer 2021) - Quiz 3
July 27, 2021

- Quiz 3 is open-notes and consists of three questions. There are 15 points available. Take a quick scan through the questions first and prioritize your time accordingly.
- Show all of your work for full marks, and ensure your notation is legible, correct, and consistent with that used in the course. Be sure to clearly distinguish between random variables and constants, and between vectors and scalars (you can write $\mathbf{X}$ as $\vec{X}$ and $\mathbf{x}$ as $\vec{x}$).
- Upload clear, legible photos/screenshots of your handwritten answers to the questions within the time window, one question at a time.
- You may refer to the lecture slides and assignments. If you need to use a result from them, either refer to it by its number or its name (if it is numbered/named), or describe the result.

Good luck!

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1. (5 points) Let $X_1, X_2, \ldots, X_n$ be a random sample from a continuous distribution with pdf

$$f_\theta(x) = \frac{1}{x \cdot \log(\theta)}, \qquad x \in (1, \theta), \quad \theta \geq e.$$

We want to test $H_0 : \theta = e$ versus $H_A : \theta \neq e$ using an LRT, which rejects $H_0$ when $\lambda(\mathbf{x}) \leq c$ for some $c \in (0, 1)$.

(a) (3 points) Given that the MLE of $\theta$ is $\hat{\theta}(\mathbf{X}) = X_{(n)}$, determine the LRT statistic $\lambda(\mathbf{X})$, simplifying your answer as much as possible. It should include an indicator function somewhere.

(b) (1 point) Show that under $H_0$, the cdf of $X_{(n)}$ is given by $F_{X_{(n)}}(x) = \begin{cases} 0, & x \leq 1 \\ \log(x)^n, & x \in (1, e) \\ 1, & x \geq e \end{cases}$

(c) (1 point) Find $c$ to make the LRT a size-$\alpha$ test.

2. (5 points) Answer each of the following questions by writing YES or NO (1 point), and justify your answer in *at most* three sentences (1.5 points).

   (a) (2.5 points) Suppose that $T(\mathbf{X})$ is sufficient for $\theta$, and that $\lambda(\mathbf{X})$ and $\lambda^*(T(\mathbf{X}))$ are the LRT statistics based on $\mathbf{X}$ and $T(\mathbf{X})$ respectively. If an LRT that rejects $H_0$ when $\lambda^*(T(\mathbf{X})) \leq 0.261$ is a size-$\alpha$ test, is an LRT that rejects $H_0$ when $\lambda(\mathbf{X}) \leq 0.261$ also a size-$\alpha$ test?

   (b) (2.5 points) You design a level-0.05 hypothesis test, observe some data $\mathbf{X} = (x_1, x_2, \ldots, x_n)$ and calculate a $p$-value of $p(x_1, x_2, \ldots, x_n) = 0.06$, and fail to reject $H_0$ at the 0.05-significance level. Your colleague notices that if you throw away $x_n$ and redo the calculations, you'll get $p(x_1, x_2, \ldots, x_{n-1}) = 0.04$. Should you reject $H_0$ at the 0.05-significance level?

3. (5 points) Suppose we have a one-sided size-$\alpha$ $Z$-test of $H_0 : \mu \geq \mu_0$ versus $H_A : \mu < \mu_0$.

   (a) (3 points) Prove that the power of the test at any parameter $\mu < \mu_0$ increases with the sample size $n$.

   (b) (2 points) Briefly explain what this result means in terms of Type I errors and/or Type II errors (if anything).