University of Toronto
STA261 (Summer 2021) - Quiz 2
July 20, 2021

- Quiz 2 is open-notes and consists of three questions. There are 15 points available. Take a quick scan through the questions first and prioritize your time accordingly.
- Show all of your work for full marks, and ensure your notation is legible, correct, and consistent with that used in the course. Be sure to clearly distinguish between random variables and constants, and between vectors and scalars (you can write $\mathbf{X}$ as $\vec{X}$ and $\mathbf{x}$ as $\vec{x}$).
- Upload clear, legible photos/screenshots of your handwritten answers to the questions within the time window, one question at a time.
- You may refer to the lecture slides and assignments. If you need to use a result from them, either refer to it by its number or its name (if it is numbered/named), or describe the result.

Good luck!

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1. (5 points) Let $X_1, X_2, \ldots, X_n$ be a random sample from a discrete distribution with pmf

$$p_\theta(x) = \frac{x^{x-1}}{x!}\theta^{x-1}e^{-\theta x}, \qquad x \in \{1, 2, 3, \ldots\}, \quad \theta \in (0,1).$$

   (a) (3 points) Find the MLE of θ assuming $\sum_{i=1}^n X_i > n$.

   (b) (1 point) Under the same assumption, calculate the Fisher information of θ given that $\operatorname{Var}_\theta(X_i) = \frac{\theta}{(1-\theta)^3}$.

   (c) (1 point) The MLE doesn't exist when $\sum_{i=1}^n X_i = n$. Calculate the probability of this event. *Hint:* look at the support of X.

2. (5 points) Answer each of the following questions by writing YES or NO (1 point), and justify your answer in *at most* three sentences (1.5 points).

   (a) (2.5 points) Would it be right to say that the observed Fisher information can't be calculated in real life because we don't know the true value of θ?

   (b) (2.5 points) Suppose $T(\mathbf{X})$ is a complete sufficient statistic which is unbiased for $\tau(\theta)$, and suppose that $V(\mathbf{X})$ is efficient for $\tau(\theta)$. Must it be true that $\mathbb{P}_\theta\left(V(\mathbf{X}) = T(\mathbf{X})\right) = 1$?

3. (5 points) Suppose that $X \sim f_\theta$ is a sample of size 1, and that $T(X) = X$ is unbiased for θ. Let that $U(X) = X^2$. You don't need to write down a single integral for this question.

   (a) (2.5 points) Show that if X is constant, then $U(X)$ is unbiased for θ^2.

   (b) (2.5 points) Show that if $U(X)$ is unbiased for θ^2, then X is constant.