University of Toronto
STA261 (Summer 2021) - Final Assessment
August 19, 2021

- This final assessment is open-notes and consists of five questions worth a total of 90 points, plus an additional bonus question worth a total of 10 points. Take a quick scan through the questions first and prioritize your time accordingly. It's recommended that you do not attempt the bonus question until you're satisfied with the rest of your work.
- Show all of your work for full marks, and ensure your notation is legible, correct, and consistent with that used in the course. Be sure to clearly distinguish between random variables and constants, and between vectors and scalars (you can write $\mathbf{X}_n$ as $\vec{X}_n$ and $\mathbf{x}_n$ as $\vec{x}_n$).
- Upload clear, legible photos/screenshots of your handwritten answers to the questions within the time window, one question at a time.
- You may refer to your own notes and the STA261 material on Quercus. If you need to use a result from the lectures or the assignments, either refer to it by its number or its name (if it is numbered/named), or describe the result.

Good luck!

The following information is relevant for Questions 1-4 and Question 6, but *not* for Question 5. Let $X_1, X_2, \ldots, X_n$ be a random sample from a continuous distribution with pdf

$$f_\theta(x) = \sqrt{\frac{\theta}{2\pi x}} e^{-\frac{\theta x}{2}}, \quad x > 0, \quad \theta > 0.$$

Let $\tau(\theta) = 1/\theta$ and let $T(\mathbf{X}) = \overline{X}_n$.

You may use the following facts without proof:

- **Fact 1**: The Gamma(α, β) distribution has pdf $\frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$ for $x > 0$, and $\alpha, \beta > 0$. Its mean is given by α/β, its variance is given by α/β^2, and its mode is given by $(\alpha-1)/\beta$ for $\alpha > 1$.
- **Fact 2**: $\int_0^\infty t^\alpha e^{-\beta t}\, dt = \frac{\Gamma(\alpha+1)}{\beta^{\alpha+1}}$ for any $\alpha > -1$ and $\beta > 0$. The gamma function satisfies $\Gamma(x+1) = x \cdot \Gamma(x)$ for $x > 0$, and $\Gamma(1/2) = \sqrt{\pi}$.
- **Fact 3**: The pdf of a $\chi^2_{(k)}$ distribution is given by $\frac{2^{-k/2}}{\Gamma(k/2)} x^{k/2-1} e^{-x/2}$.
- **Fact 4**: If $X_n \xrightarrow{d} X$ and $\mathbb{E}[|X|] < \infty$, then $\mathbb{E}[X_n] \xrightarrow{n\to\infty} \mathbb{E}[X]$.

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1. (a) Show that f_θ is in an exponential family.

(b) Show that $T(\mathbf{X})$ is unbiased for $\tau(\theta)$.

(c) Explain why $T(\mathbf{X})$ must be the UMVUE of $\tau(\theta)$.

(d) Your friend Deb claims that $T(\mathbf{X})/X_1$ is independent of $T(\mathbf{X})$. Is Deb right? Answer YES or NO and justify your answer in at most three sentences.

2. (a) Show that $\theta_{\text{MOM}} = 1/T(\mathbf{X})$.

(b) Show that $\theta_{\text{MLE}} = 1/T(\mathbf{X})$.

(c) Using your previous work, show that the Fisher information is $I_n(\theta) = n/2\theta^2$, and calculate the observed Fisher information $J_n(\mathbf{X})$.

(d) Determine the asymptotic distribution of $\hat{\theta}_{\text{MLE}}$. If your asymptotic variance has an n in it, you'll get a 0 on this sub-question.

3. Let $G_\theta(t)$ be the cdf of $T(\mathbf{X})$, and let $G_\theta^{-1}(t)$ be its inverse. Use the Neyman-Pearson lemma to derive a level-α UMP test for testing $H_0 : \theta = 4$ versus $H_A : \theta = 2$.

4. (a) Show that the $\{\text{Gamma}(\alpha, \beta) : \alpha > 0, \beta > 0\}$ family is conjugate to $\{f_\theta : \theta > 0\}$.

(b) You decide that a Gamma$(1, \beta)$ prior on θ is appropriate, but you're not sure how to choose β. Luckily, some experts in the field inform you that they're $(1 - 1/e) \cdot 100\%$ sure that θ is between 0 and 1. What choice of β is consistent with this information? (*Hint*: start by writing out the prior density in full).

(c) Instead of the prior in part (b), you decide to place Jeffreys' prior on θ. Show that Jeffreys' prior is given by $\pi^J(\theta) \propto \frac{1}{\theta}$, state whether it's proper or improper, and identify the posterior distribution based on it.

(d) Determine the MAP when using Jeffreys' prior, and show that it's consistent for the "true" θ_0, in the frequentist sense.

5. (a) Suppose that in a statistical model the MLE $\hat{\theta}(\mathbf{X})$ is unique, and that $T(\mathbf{X})$ is sufficient for θ. Prove that $\hat{\theta}(\mathbf{X})$ depends on $\mathbf{X}$ only through $T(\mathbf{X})$.

(b) Prove that if a sequence of estimators W_n for θ is asymptotically efficient, then W_n must also be asymptotically unbiased for θ.

(c) Suppose you choose a prior which is degenerate at some $\theta_0 \in \Theta$. Prove that $\{\theta_0\}$ is the only $(1-\alpha)$-HPD region for θ.

(d) Briefly explain why anyone would want to use an improper prior.

6. (a) In the context of Question 1, derive the pdf of $T(\mathbf{X})$.

(b) A twice-differentiable function defined on a (possibly infinite) interval is said to be *convex* if its second derivative is non-negative on on that interval. *Jensen's inequality* states that $\varphi(\mathbb{E}[X]) \leq \mathbb{E}[\varphi(X)]$ for any convex function $\varphi(\cdot)$, where equality holds if and only if $\varphi(x) = ax + b$ for some $a, b \in \mathbb{R}$.

In the context of Question 2, prove that $\hat{\theta}_{\text{MLE}}$ cannot be unbiased for θ.