STA2311 (Fall 2023) - Practice Problems for Class 8 (MCMC Tuning and Diagnostics)

1. Show that from a Metropolis-Hastings sampler which samples from a d-dimensional target $\pi(\boldsymbol{x})$, one can obtain samples from the j'th marginal component of π by extracting the x_j-subchain of the original samples.

2. Show that the random scan version of the Gibbs sampler is reversible.

3. Let $(X, Y, Z) \sim \pi$. We saw in Class 7 that a blocked Gibbs sampler allows us to sample $(X, Y) \mid Z$ and then $Z \mid (X, Y)$, since the stationary distribution π is preserved.

   (a) Consider the following alternative update scheme:

   i. Draw $(X', Z') \mid Y \sim \pi(x, z \mid Y)$

   ii. Draw $Y' \mid (X', Z') \sim \pi(y \mid X', Z')$

   iii. Draw $(Y'', Z'') \mid X' \sim \pi(y, z \mid X')$

   Show that the update $(X, Y, Z) \to (X', Y'', Z'')$ also preserves the stationary distribution π.

   (b) Consider the update scheme obtained by simply skipping the second step above:

   i. Draw $(X', Z') \mid Y \sim \pi(x, z \mid Y)$

   ii. Draw $(Y'', Z'') \mid X' \sim \pi(y, z \mid X')$

   Show that the update $(X, Y, Z) \to (X', Y'', Z'')$ is still valid.

4. Consider a 4-component Gibbs sampler for sampling $(X, Y, Z, W) \sim \pi$. Suppose it is possible to sample from $\pi(y \mid X, Z)$ and $\pi(Z \mid X, Y)$.

   (a) Consider the following update scheme:

   i. Draw $W' \mid (X, Y, Z) \sim \pi(w \mid X, Y, Z)$

   ii. Draw $X' \mid (Y, Z, W') \sim \pi(x \mid Y, Z, W')$

   iii. Draw $Y' \mid (X', Z) \sim \pi(y \mid X', Z)$

   iv. Draw $Z' \mid (X', Y') \sim \pi(z \mid X', Y')$

   Show that the update $(X, Y, Z, W) \to (X', Y', Z', W')$ does *not* preserve the stationary distribution π.

   (b) Consider the following update scheme:

   i. Draw $W' \mid (X, Y, Z) \sim \pi(w \mid X, Y, Z)$

   ii. Draw $X' \mid (Y, Z, W') \sim \pi(x \mid Y, Z, W')$

   iii. Draw $(W'', Y') \mid (X', Z) \sim \pi(y, w \mid X', Z)$

   iv. Draw $(W''', Z') \mid (X', Y') \sim \pi(w, z \mid X', Y')$

   Show that the update $(X, Y, Z, W) \to (X', Y', Z', W''')$ *does* preserve the stationary distribution π.

5. Consider the 2-dimensional target distribution

$$\pi(x, y) \propto \frac{1}{2} f(x, y) + \frac{1}{2} f(-x, -y),$$

   where

$$f(x, y) \propto \exp\left(-\frac{1}{2} \left(8x^2y^2 + x^2 + y^2 + 40xy - 8x - 8y \right) \right).$$

   (a) Construct an MCMC algorithm to sample from π.

   (b) Estimate $\mathbb{E}_\pi[\sin(X) \cdot \cos(Y)]$ and construct a 95% confidence interval for your estimator.