STA2311 (Fall 2023) - Practice Problems for Class 6 (Simulation and Monte Carlo)

1. Prove two results mentioned in the antithetic variates setup:

   (a) If $g_1, g_2 : \mathbb{R} \to \mathbb{R}$ are monotone functions, then $\text{Cov}(g_1(X), g_2(X)) \geq 0$ for any random variable X.

   (b) Prove that the antithetic variates estimator is a special case of the control variates estimator.

2. Suppose we are interested in sampling from the Gamma(α, β) distribution with density

$$f(x|\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \cdot \mathbb{1}_{x>0},$$

   with $\alpha, \beta > 0$. For this problem you may assume that $\Gamma(\alpha)$ can be computed for any $\alpha > 0$.

   (a) If $\alpha = k$ is a positive integer and $X_1, \ldots, X_k \overset{iid}{\sim} \text{Exp}(\beta)$ (with density $f(x|\beta) = \beta e^{-\beta x} \cdot \mathbb{1}_{x>0}$), then $\sum_{i=1}^k X_i \sim \text{Gamma}(k, \beta)$. Using this result, how do you propose to sample from the Gamma(k, β) distribution?

   (b) Using the fact that it is thus relatively easy to sample from the Gamma(α, β) distribution when α is an integer, provide an efficient accept-reject estimator to sample from Gamma(α, β) for any $\alpha > 1$.

3. Let $\boldsymbol{U}_1, \ldots, \boldsymbol{U}_k \overset{iid}{\sim} \text{Unif}([0,1]^d)$ with $\boldsymbol{U}_i = (U_{i1}, \ldots, U_{id})$, and let $\tau_1, \ldots, \tau_d$ random permutations of $(0, \ldots, k-1)$ which are independent of the $\boldsymbol{U}_i$'s. Define

$$X_{ij}^{(1)} = \frac{\tau_j(i-1) + U_{ij}}{k}, \quad 1 \leq i \leq k, \quad 1 \leq j \leq d.$$

   (a) Show that $X_{ij}^{(1)} \sim \text{Unif}(0,1)$.

   (b) Determine $\text{Corr}(X_{ij}^{(1)}, X_{i'j}^{(1)})$ for $i \neq i'$.

   (c) For $r \geq 2$, let $\tau_1^{(r)}, \ldots, \tau_d^{(r)}$ be new random permutations of $(0, \ldots, k-1)$ and recursively define

$$X_{ij}^{(r)} = \frac{\tau_j^{(r)}(i-1) + X_{ij}^{(r-1)}}{k}, \quad 1 \leq i \leq k, \quad 1 \leq j \leq d.$$

   Determine $\text{Corr}(X_{ij}^{(r)}, X_{i'j}^{(r)})$ for $i \neq i'$. *Hint: do it first for* $r = 2, 3, \ldots$.

4. Let X be an $\mathbb{R}$-valued random variable with density proportional to $\exp(-|x|^3/3)$. Consider estimating $\sigma^2 = \mathbb{E}[X^2]$.

   (a) Estimate σ^2 using importance sampling.

   (b) Estimate σ^2 using rejection sampling.

5. Consider a Poisson regression model in which $Y \mid X \sim \text{Poisson}(e^{\lambda(X)})$, where $\lambda(x) = \alpha_0 + \alpha_1 x$. Suppose we plan a simulation study in which we know the values of α_0, α_1, and also that the marginal distribution of X is $\mathcal{N}(0,1)$. Design an algorithm to sample from the conditional distribution of X given a particular value of $Y = y$.

6. Suppose that $Y_1, \ldots, Y_N$ is a sample produced by rejection sampling based on a target f and proposal g, with $M = \sup(f/g)$. Suppose that $X_1, \ldots, X_t$ is the accepted sub-sample and $Z_1, \ldots, Z_{N-t}$ is the rejected subsample. The goal is to estimate some $I = \mathbb{E}_f[h]$.

   (a) Show that both $\delta_1 = \frac{1}{t}\sum_{i=1}^{t} h(X_i)$ and

   $$\delta_2 = \frac{1}{N-t}\sum_{i=1}^{N-t} h(Z_i) \cdot \frac{(M-1)f(Z_i)}{M \cdot g(Z_i) - f(Z_i)}$$

   are both unbiased estimators of I when $N > t$.

   (b) Show that δ_1 and δ_2 are independent.

   (c) Determine the optimal (in terms of minimal variance) weight β^* in the weighted estimator $\delta_3 = \beta\delta_1 + (1-\beta)\delta_2$.

7. Suppose we are Bayesians, and we wish to compare two posterior distributions characterized by densities $\pi_1(\theta) = \tilde{\pi}_1(\theta)/c_1$ and $\pi_2(\theta) = \tilde{\pi}_2(\theta)/c_2$ with the same support, for which only the $\tilde{\pi}_i$'s can be computed, and for which the normalizing constants c_1, c_2 are unknown. The goal here is to estimate $r := c_1/c_2$.

   (a) Show that r can be approximated by

   $$\frac{1}{n}\sum_{i=1}^{n} \frac{\tilde{\pi}_1(\theta_i)}{\tilde{\pi}_2(\theta_i)}, \qquad \theta_1, \theta_2, \ldots, \theta_n \overset{iid}{\sim} \pi_2.$$

   (b) Show that

   $$\frac{\int \tilde{\pi}_1(\theta) \cdot \alpha(\theta) \cdot \pi_2(\theta)\, \mathrm{d}\theta}{\int \tilde{\pi}_2(\theta) \cdot \alpha(\theta) \cdot \pi_1(\theta)\, \mathrm{d}\theta} = r$$

   for any function $\alpha(\cdot)$ such that both integrals are finite, and in particular we have the interesting identity

   $$\frac{\mathbb{E}_{\pi_2}[\tilde{\pi}_2^{-1}]}{\mathbb{E}_{\pi_1}[\tilde{\pi}_1^{-1}]} = r.$$

   Thus explain why

   $$\hat{R}_\alpha := \frac{\sum_{i=1}^{n} \tilde{\pi}_1(\theta_{2i}) \cdot \alpha(\theta_{2i})}{\sum_{i=1}^{n} \tilde{\pi}_2(\theta_{1i}) \cdot \alpha(\theta_{1i})}, \qquad \theta_{1i} \overset{iid}{\sim} \pi_1, \qquad \theta_{2i} \overset{iid}{\sim} \pi_2$$

   is a convergent estimator of r as $n \to \infty$.

   (c) One can show that the *relative MSE* of $\hat{R}_\alpha$, defined by $\mathbb{E}[(\hat{R}_\alpha - r)^2]/r^2$, is given by

   $$\frac{1}{2n}\frac{\int \tilde{\pi}_1(\theta) \cdot \tilde{\pi}_2(\theta) \cdot (\tilde{\pi}_1(\theta) + \tilde{\pi}_2(\theta)) \cdot \alpha(\theta)^2\, \mathrm{d}\theta}{\left(\int \tilde{\pi}_1(\theta) \cdot \tilde{\pi}_2(\theta) \cdot \alpha(\theta)\, \mathrm{d}\theta\right)^2} - \frac{2}{n}.$$

   Show that the relative MSE is minimized by taking

   $$\alpha(\theta) \propto \frac{1}{\tilde{\pi}_1(\theta) + \tilde{\pi}_2(\theta)}.$$

   Hint: use the Cauchy-Schwarz inequality.