STA201 Week 13: Game Theory

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Last Week

- Credit Scores
- Credit Risk
- 2008 Financial Crisis

This Week

What Is Game Theory?

Definition

Game theory is the study of mathematical models of strategic interaction between rational decision-makers.

- Game theory applies to a wide range of behavioural relations, and is now an umbrella term for the **science of logical decision-making** in humans, animals, and computers (Wikipedia)
- It is the study of strategic interdependence (where different actors' decisions affect each other's welfare) – hence "actors need to anticipate, act, and react" (Spaniel)
- Like much of probability and statistics, game theory was originally inspired by a gambling problem

The Waldegrave Problem I

- The Waldegrave Problem (early 18^{th} century) – actually several problems
- How to play:
  - Standard deck of cards: kings high, aces low
  - The Dealer deals one card each to herself and to the Receiver (face down)
  - The Dealer and the Receiver each see the value of their own card, but not the other
  - The Receiver may switch cards with the Dealer, unless the Dealer has a king
  - The Dealer may choose to switch her card with a card drawn randomly from the deck (unless the drawn card is a king, in which case she must keep her original card)
  - Whoever holds the higher card wins the game, and the Dealer wins any ties

The Prisoner's Dilemma I

Research ANd Development

- The classic prototype of a game theory problem (originated in 1950 at RAND as part of military research on strategy)
- Setup: Alice and Bob decide to rob a jewelry store, but are arrested as they approach the back door with their crowbars, and placed in separate interrogation rooms so they can't communicate with each other

The Prisoner's Dilemma II

- The police give each of them the following information:
  - They can charge you both with trespassing (1-month sentence)
  - There's not enough evidence to charge you both with breaking and entering (10-month sentence) – they need your testimony!
  - If you confess and provide the testimony, you will go free and your accomplice will serve the full 10 months
  - If you both confess, you'll both serve 5 months each
  - Your accomplice is receiving the same offer
- **We assume for simplicity** that Alice and Bob are both sociopaths and only care about minimizing their own jail time; neither cares about the other
  - Be careful about assumptions in analyzing your game!
  - More complicated games may have different assumptions, including multiple goals for the other players

The Prisoner's Dilemma III

PollEverywhere:

If you were Alice, what would you do?

- Fess up 69%
- Keep your mouth shut! 31%

The Prisoner's Dilemma IV

- Decisions like these are best analyzed using a *payoff matrix*:

Bob's choices

(Alice # months, Bob # months)

Alice's choices	Clam up	Confess
Clam up	(1, 1)	(10, 0)
Confess	(0, 10)	(5, 5)

- If Alice knows Bob will clam up, what should she do? Confess!
- If Alice knows Bob will confess, what should she do? Confess!
- If Bob knows Alice will clam up, what should he do? Confess!
- If Bob knows Alice will confess, what should he do? Confess!

Check!

The Prisoner's Dilemma V

- The previous analysis might suggest that both Alice and Bob should confess and serve 5 months each
- How does that compare to the result if they both clam up? It's worse!

Definition

Strategy X **strictly dominates** Strategy Y for a player if Strategy X provides a greater payoff for that player than Strategy Y, *regardless of what the other players do* (Spaniel)

- It turns out that the best *overall* strategy in this type of game is always different from the strictly dominant strategy
- In the Prisoner's Dilemma, the strictly dominant strategy for both Alice and Bob would be to confess (hoping the other clams up)
- Both clamming up would result in the fewest months of prison time overall

The Prisoner's Dilemma VI

- The Prisoner's Dilemma has seen many real-world applications (besides police interrogations)
- Two states hostile to each other:
  - Each country prefers peace to the destruction of war
  - But **if** if there is a war, having the first-strike advantage is very significant
  - Example: Cold War (Soviet/US nuclear arms race)
- Trade wars
  - Each country wants to export its goods without the other country imposing a tariff
  - Each country also wants to slap a tariff on goods imported from the other country

Nash Equilibrium I

John Forbes Nash, Jr. (1928-2015) ⇐ Click for scene from "A Beautiful Mind" (2001)

- 1994 Nobel Prize in Economics for game theory (shared with Reinhard Selten and John Harsanyi)

Dear Professor Lefschetz:

This is to recommend Mr. John F. Nash, Jr. who has applied for entrance to the graduate college at Princeton.

Mr. Nash is nineteen years old and is graduating from Carnegie Tech in June. He is a mathematical genius.

Yours sincerely,

Richard J. Duffin

Richard J. Duffin

Nash Equilibrium II

Definition

Nash equilibrium is a situation involving two or more players in which each player is assumed to know the equilibrium strategies of the other players, and no player has anything to gain by changing only their own strategy.

- When **no** player in the game can do better by unilaterally changing their strategy
- "Knowing the strategies of the other players, and treating the strategies of the other players as set in stone, can I benefit from changing my strategy?" If the answer is "no", then you're in Nash equilibrium

Nash Equilibrium III

Definition

Nash equilibrium is a situation involving two or more players in which each player is assumed to know the equilibrium strategies of the other players, and no player has anything to gain by changing only their own strategy.

- In a Nash equilibrium, each player's strategy is his best response to all the other strategies in that equilibrium (Wikipedia)
- In the Prisoner's Dilemma, the Nash equilibrium is the situation in which both players confess
- "No regrets" property: after the game is over, all players know that they've made the best choices they could under the circumstances

The Stag Hunt I

- An example of cooperation in games
- Described by Jean-Jacques Rousseau (1750) to illustrate the conflict between safety and social cooperation
- Setup:
  - Two hunters go out to hunt – they can choose to hunt a stag (takes longer, requires two hunters to cooperate) or hares (quick procedure, easily done alone)
  - Stags yield much more meat than hares, but if only one hunter hunts a stag, she will get no meat
  - The stag choice is **risky** but the hare choice is safe
    - If the other hunter doesn't also choose stag, you go home hungry
    - If the hunters agree to hunt a stag but one hunter reneges on the agreement, the other goes home hungry
    - With the hare choice, you always go home with something to eat - it is the so-called *risk-dominant* choice

The Stag Hunt II

- Generic payoff matrix:

(first hunter's choice, second hunter's choice)

1st choice: stag
2nd choice: hare
3rd choice: nothing

		First hunter's choice	
		Stag	Hare
Second hunter's choice	Stag	(1, 1)	(2, 3)
	Hare	(3, 2)	(2, 2)

- Here there are *two* Nash equilibrium situations: (stag, stag) and (hare, hare)
  - "The best response to stag is stag; the best response to hare is hare"
  - The (stag, stag) choice is said to be *Pareto-optimal*: everyone is better off than with (hare, hare) but its also the riskier of the two Nash equilibrium outcomes

The Stag Hunt III

- The Stag Hunt game involves issues of...
  - Future benefit (cooperative stag hunt) versus immediate reward (hare hunt)
  - Social cooperation versus self-sufficiency
  - Risk tolerance
  - Trust between the parties
- Applications:
  - Resource sharing: eg, neighboring counties share a river
  - Insurance policies: the company can choose whether or not to trust the applicant and offer a lower rate; the applicant can choose to cooperate (or not) with low-risk behaviour
  - Real estate: cooperate with the owner's desire to get the highest possible price, or suggest a lower price in order to increase the chances of a sale

Zero-Sum Games I

- Games where players benefit only at the expense of other player(s)
- Regardless of the strategy, the total benefit for all players always adds up to zero
- Examples: Chess and most other classical 2-player games
  - One winner (+1) and one loser (-1)

Consider stalemates/ties as worth 0 for each player
so the "sum" is still 0 at the end

Zero-Sum Games II

PollEverywhere: Is the Steg Hunt a zero-sum game?

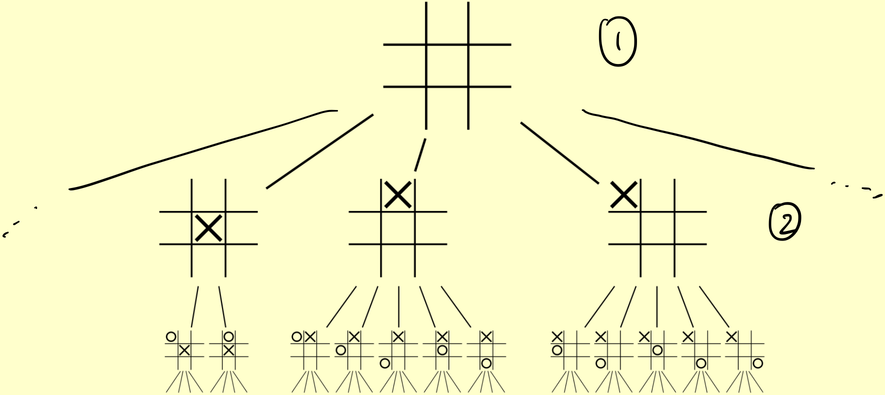
- Yes 15%
- No 85%

Sequential Games I

- Games where players don't move simultaneously, but rather in sequence
- Therefore, players know something about the other players' previous actions
- These can be represented using game trees
- Examples:
  - With perfect information: Tic-Tac-Toe, Chess, Checkers
  - With imperfect information: Poker, Bridge
  - Game of Nim

Sequential Games II

- A (partial) game tree for Tic-Tac-Toe:



Combinatorial Games

- Games where the huge number of possible strategies makes it hard to choose an optimal strategy
- Examples:
- A big field in artificial intelligence is using computers to study combinatorial games and play them using probabilistic strategies
  - Alphabeta pruning (used by *Stockfish*, the best Chess engine today)
  - Artificial neural networks (Google Deepmind's *AlphaGo* famously beat the top-ranked world champion Go player in 2016)

"AlphaGo" (2017) movie on Netflix!

Evolutionary Games I

- Games where the players adjust their strategies over time (not necessarily as rational decisions)
- Example: Biological evolution (John Maynard Smith)
    - Always a multiplayer game
    - The "strategy" here means having some genetic trait (e.g., bright plumage) eg. birds
    - The goal of the game is to reproduce as much as possible
    - The payoff is better reproductive fitness (ability to have more offpsring) than the other players
    - The mix of genetic traits in the population may shift with succeeding generations
        - Genes that increase reproductive fitness become more frequent
        - Genes that decrease reproductive fitness (e.g., dull plumage) become rarer

Evolutionary Games II

(Wikipedia)

Pure Strategies vs. Mixed Strategies

- *Mixed* means randomizing over multiple strategies (probabilistically) versus choosing a single strategy
- More complicated mathematics are needed to analyze outcomes and calculate payoffs
- For example: the Waldegrave Problem from above
  - A full analysis would involve analyzing what each player would do for each of 13 cards, which is $2^{13} \times 2^{13}$ (over 67 million) different combinations of strategies
  - We can greatly decrease our work by eliminating combinations that don't make sense (for example, hold onto a 3 but discard a 6)
  - Still leaves *a lot* of calculations, however

The Waldegrave Problem II

Guarantees the highest probabilities of winning for both the Dealer & Receiver

- It turns out that the best solution is a probabalistic one!
- Receiver should keep cards with value 8 or higher, change cards with value 6 or lower (pure strategies)

what about card #7?

- Dealer should keep cards with value 9 or higher, change cards with value 7 or lower (pure strategies)
- If Receiver gets a 7: keep card with probability $\boxed{\frac{5}{8}}$, change card with probability $\boxed{\frac{3}{8}}$ (mixed strategy)
- If Dealer gets an 8: keep card with probability $\boxed{\frac{3}{8}}$, change card with probability $\boxed{\frac{5}{8}}$ (mixed strategy)

The Waldegrave Problem III

- Would you rather be the Dealer or the Receiver?
- The Dealer has the last move and wins any ties, while the Receiver has first move
- As it turns out...
  - The probability that the Dealer wins is 0.487
  - What's the probability that the Receiver wins?

    P(Receiver wins) = 1 − P(Dealer wins) = 1 − 0.487

    = 0.513

No ties (⇒ Dealer wins)

This is a zero-sum game! (The receiver wins if and only if the dealer loses & vice versa)

Coalitions and Alliances

- In the Prisoner's Dilemma, both Alice and Bob clamming up would result in the fewest months of prison time overall
- **But** if it's in each player's individual interest to confess, his/her own strictly dominant strategy (both to confess) is likely to win out
- What if they both agreed beforehand to both clam up?

The Pirate Puzzle I

- Three pirates (Alice, Bob, and Charlie) are going to split up 100 gold pieces
- The leadership structure is such that Alice is stronger than Bob, who is stronger than Charlie
- They decide to divide up the gold according to the following rules:
  - Alice, the strongest, suggests a division (say 90 to me, 9 to Bob, 1 to Charlie). Everyone (including Alice) votes on this proposal; in case of a tie, the proposer has the casting vote
  - If they vote to agree, the split is done. If the proposal is voted down, the proposer is thrown overboard and dies
  - If the proposer is thrown overboard and dies, the next strongest pirate becomes the proposer
  - The game continues until a proposal is accepted

The Pirate Puzzle II

- First priority: Stay alive!
- Second priority: Get some gold.
- If you are Alice, what do you propose?

The Pirate Puzzle III

- Consider what will happen if Alice is thrown overboard:
  - Bob will have complete power; he can propose whatever he likes, and as proposer he can overrule Charlie's vote. So Bob can suggest 100 gold pieces to himself and nothing to Charlie.
  - Therefore Charlie has a vested interest in keeping Alice alive, theoretically for *any* offer above no gold coins at all
- Now, from Alice's point of view...
  - She only needs one vote (besides her own) for her proposal to be accepted
  - She will buy Charlie's vote with the minimum amount necessary
  - Therefore, Alice suggests 99 gold pieces for herself, 1 gold piece for Charlie, and 0 to Bob; Alice and Charlie vote in favour, and the offer is accepted
  - Alice gains tremendous power by exploiting the conflict of interest between Bob and Charlie

Further Reading/Viewing

- William Spaniel's Game Theory 101 series on YouTube (and companion book, *Game Theory 101: The Complete Textbook*)
- Presh Talwalkar has given an interpretation of the Pirate Puzzle in the opening scene of *The Dark Knight* (2008)

Challenge: Suppose there are 5 pirates $A > B > C > D > E$.
What is the highest # of coins A can end up with?