An Introduction to Hidden Markov Models

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Hidden Processes in Real Data

- Real-world time series often exhibit abrupt or gradual changes in behavior that are driven by unobserved states (i.e., latent variables)

Astronomy: Flaring and quiescence in stellar X-ray light curves
- Latent variable: flare intensity or state

Ecology: Animal movement switching between foraging and resting
- Latent variable: behavioral mode

Finance: Stock returns alternating between volatility regimes
- Latent variable: market state

Bioinformatics: Coding vs. non-coding DNA regions
- Latent variable: genomic structure

Speech: Recognizing spoken units from acoustic signals
- Latent variable: spoken unit

Enter Hidden Markov Models

- Hidden Markov models give us a structured way to model time-dependent processes whose behavior depends on a hidden state that evolves over time

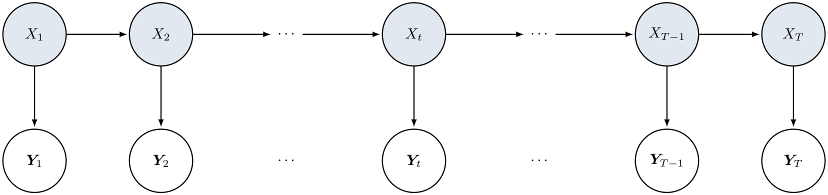
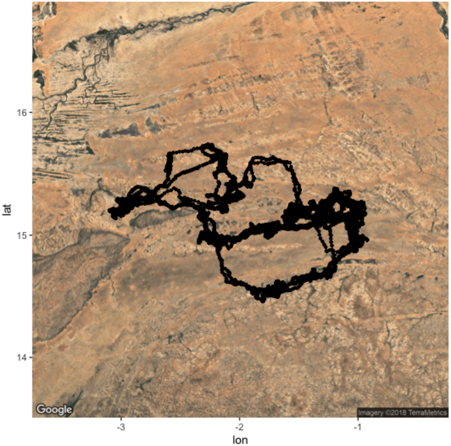


Figure: A graphical model of the standard discrete-time HMM dependence structure

Example: African Elephant Movement

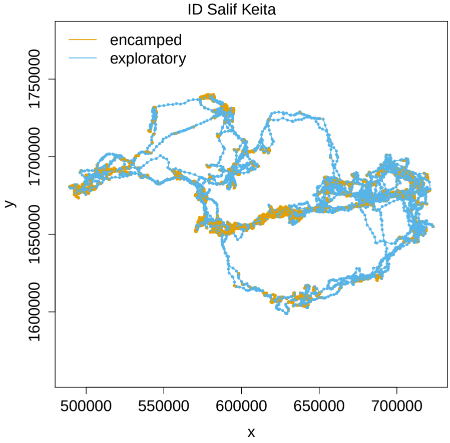
- The figure below shows an African elephant's tracks in Mali over several days [Wall et al., 2014]

- It is believed that elephants typically spend time in either of two states: *encamped* and *exploratory*



Hidden States Revealed

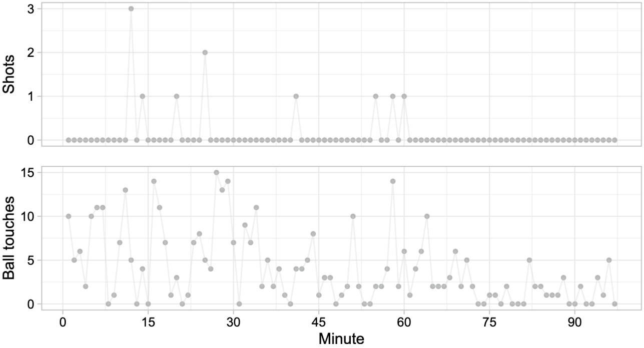
- [McClintock and Michelot, 2018] fit a 2-state HMM to the observed data, allowing ecologists to classify the elephant's state at each time point and predict its future states



Example: Momentum in Football (aka Soccer) Matches

- The figure below shows a bivariate time series of the number of shots on goal (top) and the ball touches (bottom) of Borussia Dortmund for a match vs. FC Schalke 04 [Ötting et al., 2023]

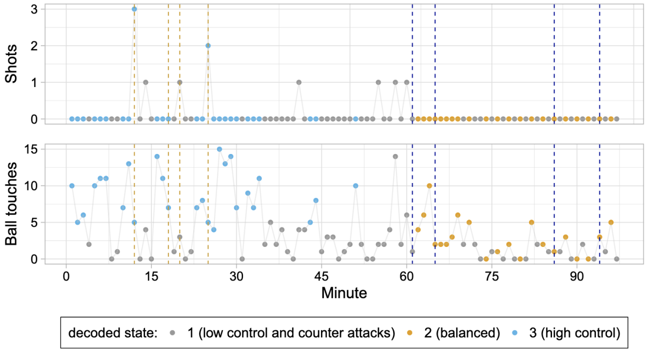
- We imagine three states for Borussia: *low control*, *balanced*, and *high control*



Hidden States Revealed

- [Ötting et al., 2023] fit a 3-state HMM to the data, with state classifications shown below

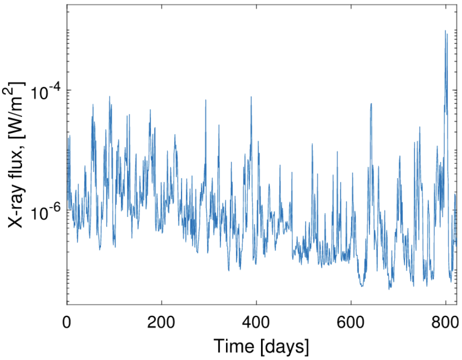
- The vertical dashed lines show goals scored by Borussia (yellow lines) and Schalke 04 (blue lines)



Example: Solar Flare Activity

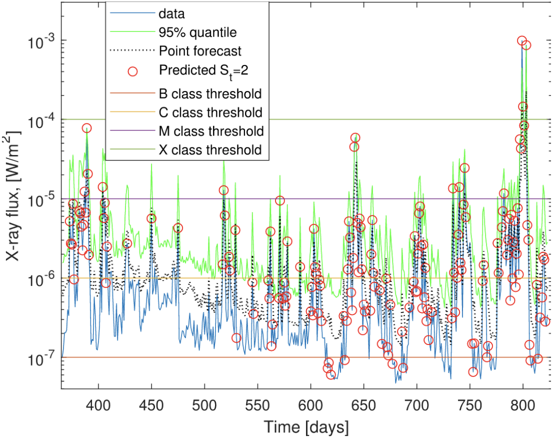
- The figure below shows solar X-ray log flux (from GOES data) in the period from 1 July 2015 to 30 September 2017 [Stanislavsky et al., 2020]

- They assume two states: *low activity* ("1") and *high activity* ("2")



Hidden States Revealed... and Predicted!

- [Stanislavsky et al., 2020] fit a 2-state HMM to rolling 365 day windows of the data, and predict both the solar flux and the state for the following day



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Mixture Models

- Let $X \in \mathcal{X}$ be a random variable with pdf $\pi(x)$ or pmf $\pi_x = \mathbb{P}(X = x)$
- Conditional on $X = x$, let $Y \in \mathcal{Y}$ be a random variable with pdf/pmf $f_x(y)$
- The *unconditional* pdf/pmf of Y is given by

$$f(y) = \int_{\mathcal{X}} \pi(x) \cdot f_x(y) \, \mathrm{d}x \quad \text{or} \quad f(y) = \sum_{x \in \mathcal{X}} \pi_x \cdot f_x(y)$$

  and Y is said to follow a **mixture model**

- Mixture models have a simple design that can accommodate unobserved heterogeneity in a population
- They are often used to handle multi-modal distributions

Special Case: Finite Mixture Models

- When $\mathcal{X} = \{1, 2, \ldots, K\}$, we have a $\boldsymbol{K}$**-component finite mixture model** with pdf/pmf

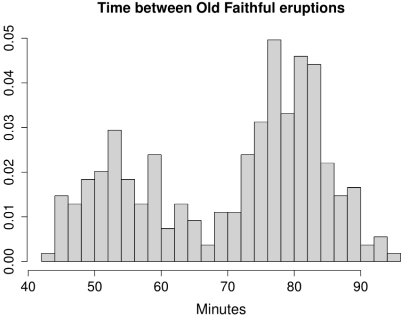
$$f(y) = \sum_{k=1}^{K} \pi_k \cdot f_k(y)$$

- Note: in general, each $f_x(y)$ can — and usually does — have an associated vector of parameters $\boldsymbol{\theta}_x$ that varies with x

- We often write $f_x(y; \boldsymbol{\theta}_x)$ to emphasize dependence on the state-dependent parameter $\boldsymbol{\theta}_x$

Example: Time Between Old Faithful Eruptions

- The figure below shows a histogram of time between eruptions for the Old Faithful geyser in Yellowstone National Park, Wyoming, USA [Azzalini and Bowman, 1990]
- The observations seem to include two distinct components
- Histograms like this are highly characteristic of finite mixture models



Maximum Likelihood for Finite Mixture Models

- Given an independent sample $y_1, \ldots, y_n \overset{iid}{\sim} f$, the likelihood function is given by

$$L(\boldsymbol{\theta}, \boldsymbol{\pi} \mid y_{1:n}) = \prod_{i=1}^{n} \left(\sum_{k=1}^{K} \pi_k \cdot f_k(y_i; \boldsymbol{\theta}_k) \right)$$

  with $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_K)$ and $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_K)$
  - ...and the log-likelihood by $\ell(\boldsymbol{\theta}, \boldsymbol{\pi} \mid y_{1:n}) = \sum_{i=1}^{n} \log\left(\sum_{k=1}^{K} \pi_k \cdot f_k(y_i; \boldsymbol{\theta}_k)\right)$

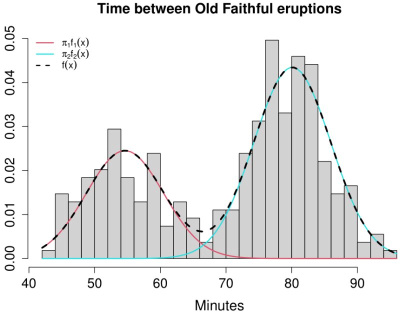
- Numerical maximization (or often the *EM algorithm*) can be used to obtain the MLEs of $\boldsymbol{\theta}$ and $\boldsymbol{\pi}$

- If some/all f_k are in the same parametric family, it is good practice to somehow (e.g., by imposing order constraints) identify the parameters of the model to prevent label switching

Back to Old Faithful

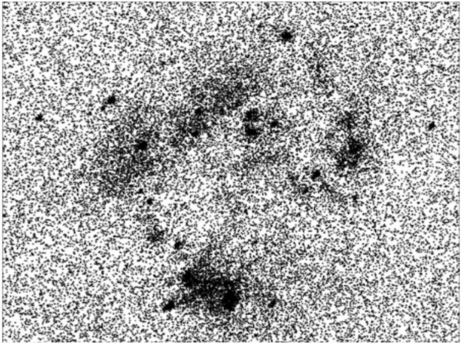
- Suppose we assume a 2-component Gaussian mixture model (i.e., $K = 2$ and each f_k is a univariate Gaussian pdf)

- If we perform maximum likelihood estimation, we get that
  - $f_1(y)$ is estimated to be $\mathcal{N}(54.6, 5.9^2)$
  - $f_2(y)$ is estimated to be $\mathcal{N}(80.1, 5.9^2)$
  - π_1 is estimated to be 0.36 (thus π_2 is estimated as 0.64).



Finite Mixture Models in Astronomy: Stellar Populations

- Astronomical populations often consist of overlapping groups (e.g., stars in different evolutionary phases)
- Finite mixture models help disentangle these subpopulations using photometric data [Fan et al., 2023]

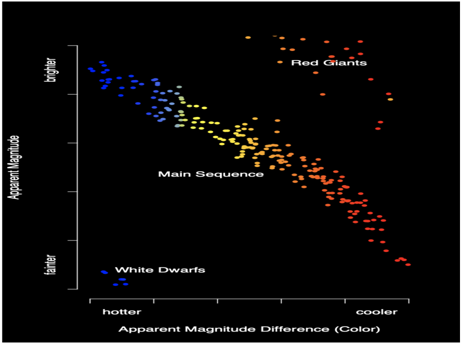


Chandra X-ray observations of colliding Antennae galaxies; the source appears over a diffuse background

Hubble optical image of colliding Antennae galaxies; emission sources are spatially structured (image credit: NASA, ESA, and the Hubble Heritage Team)

Finite Mixture Models in Astronomy: Source Separation

- Finite mixture models group spatial or photometric patterns
- We will see that HMMs extend this idea to sequences, where latent group membership evolves over time



Stylized color–magnitude diagram; mixture components reflect stellar evolution stages

Hubble optical image of the Pleiades cluster; mixture models separate cluster members from the background (image credit: NASA, ESA and AURA/Caltech

Markov Chains

- A **discrete time Markov chain** on $\mathcal{X}$ is an $\mathcal{X}$-valued stochastic process $\{X_t\}$[1] that satisfies the Markov property:

$$\mathbb{P}(X_{t+1} \in A \mid X_t = x_t, \dots, X_1 = x_1) = \mathbb{P}(X_{t+1} \in A \mid X_t = x_t)$$

  for $A \subseteq \mathcal{X}$ and $t \geq 0$
  - i.e., the distribution of X_{t+1} is entirely determined by X_t

- A discrete time Markov chain on $\mathcal{X}$ is fully characterized by
  1. An initial pdf $\delta(x)$ or pmf $\delta_x = \mathbb{P}(X_0 = x)$ that determines the distribution of X_0
  2. A transition pdf $\gamma^{(t)}(x_{t-1}, x)$ or pmf $\gamma^{(t)}_{x_{t-1},x} = \mathbb{P}(X_t = x \mid X_{t-1} = x_{t-1})$ that determines the conditional distribution of X_t given $X_{t-1} = x_{t-1}$

- If the transition pdf/pmf does not depend on t, then the chain is said to be **time-homogeneous**

[1] Notation: $\{X_t\}$ means the (possibly infinite) sequence $X_0, X_1, X_2, \dots$

Finite Space Markov Chains: Transition Probabilities

- An important special case is a time-homogeneous Markov chain on $\mathcal{X} = \{1, 2, \ldots, K\}$

- Here, the transition probability $\gamma_{i,j}$ (no superscript!) is the probability that the chain enters state j at time $t+1$ given that it is in state i at time t

- We can collect the K^2 transition probabilities into a **transition probability matrix**

$$\mathbf{\Gamma} = \begin{pmatrix} \gamma_{1,1} & \cdots & \gamma_{1,K} \\ \vdots & \ddots & \vdots \\ \gamma_{K,1} & \cdots & \gamma_{K,K} \end{pmatrix}$$

- One can show that unconditional probability $\mathbb{P}(X_t = k)$ is given by the kth entry of $\boldsymbol{\delta}\mathbf{\Gamma}^t$, where $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_K)$

Markov Chains: Stationary and Limiting Distributions

- A Markov chain has a **limiting distribution** if the distribution of X_t (starting from any initial distribution) exists as $t \to \infty$

- A time-homogeneous Markov chain is said to have a **stationary distribution** if there exists a pdf $s(x)$ or a pmf s_x which satisfies

$$\int_{\mathcal{X}} s(x) \cdot \gamma(x, x') \, \mathrm{d}x = s(x') \quad \text{or} \quad \sum_{x \in \mathcal{X}} s_x \cdot \gamma_{x,x'} = s_{x'}$$

  - In the finite space case, if $\boldsymbol{s} = (s_1, \ldots, s_K)$, then the first statement is equivalent to $\boldsymbol{s}\boldsymbol{\Gamma} = \boldsymbol{s}$

- A stationary distribution exists under mild conditions, and when it does it is *equal to the limiting distribution* (and hence unique)

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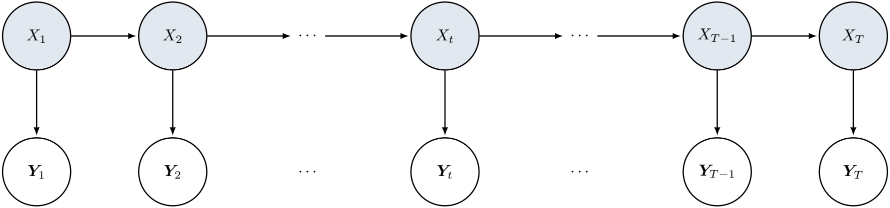
Serial Dependence

- We now consider an observed time series $\{\boldsymbol{Y}_t\}$
- Such time series commonly exhibit dependence between consecutive observations — a phenomenon known as **serial dependence**
- But sometimes, this serial dependence reflects a deeper structure: what if the behavior of $\boldsymbol{Y}_t$ is driven by an unobserved **state process** $\{X_t\}$?
- In particular, what if...
  - $\{X_t\}$ evolves as a Markov chain, and
  - The distribution of $\boldsymbol{Y}_t$ depends on the current state X_t? That is, the statistical properties of the observed process change over time depending on the hidden state

Putting Things Together: the HMM

- This generative structure informally defines a hidden Markov model



- The unobserved state process $\{X_t\}$ (shaded nodes) is a Markov chain
- The observed process $\{\boldsymbol{Y}_t\}$ (clear nodes) is conditionally independent given the states: each $\boldsymbol{Y}_t$ depends only on X_t

The HMM: General Definition

- A **discrete time hidden Markov model (HMM)** consists of...
  1. A latent process $\{X_t\}$ evolving as a Markov chain on some state space $\mathcal{X}$
     - Initial pdf/pmf $\delta(x)$
     - A transition pdf/pmf $\gamma^{(t)}(x_{t-1}, x)$
  2. An observation process $\{\boldsymbol{Y}_t\}$ on a space $\mathcal{Y}$ which is conditionally independent given the states:[2]

$$\mathbb{P}\Big(\boldsymbol{Y}_t \in A \mid X_{1:T}, \boldsymbol{Y}_{1:(t-1)}\Big) = \mathbb{P}(\boldsymbol{Y}_t \in A \mid X_t)$$

  3. A state-dependent distribution model:

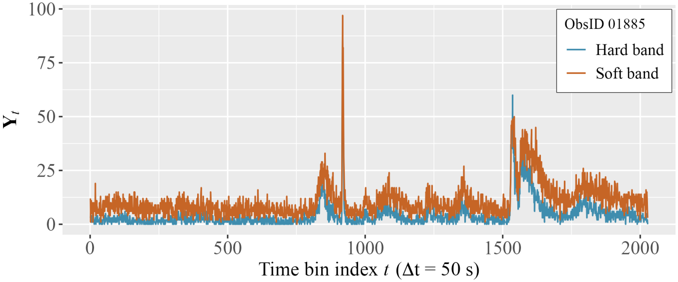
$$\boldsymbol{Y}_t \mid X_t = x \sim f_x$$

- Such an HMM is fully characterized by
  1. The initial pdf $\delta(x)$ or pmf $(\delta_x)_{x \in \mathcal{X}}$
  2. The transition pdf $\gamma^{(t)}(x_{t-1}, x)$ or pmf $\gamma^{(t)}_{x_{t-1},x} = \mathbb{P}(X_t = x \mid X_{t-1} = x_{t-1})$
  3. The state-dependent pdfs/pmfs $\{f_x : x \in \mathcal{X}\}$

[2] Notation: $X_{1:t}$ means $(X_1, X_2, \ldots, X_t)$ and similarly for $\boldsymbol{Y}_{1:t}$

Example: Flaring Behaviour of EV Lac

- [Zimmerman et al., 2024] study X-ray light curves of the red dwarf star EV Lac
- The figure below shows photon counts in soft and hard bands for EV Lac over several days



Example: Flaring Behaviour on EV Lac

- [Zimmerman et al., 2024] use a univariate Poisson state-space model to the capture flaring behaviour

- The latent Markov chain $\{X_t\}$ evolves as an AR(1) process:

$$X_t = \phi X_{t-1} + \epsilon_t, \quad \epsilon_t \sim \mathcal{N}(0, \sigma^2)$$

- The observed process $\{\boldsymbol{Y}_t = (Y_{t,1}, Y_{t,2})\}$ is a 2-tuple of soft and hard band X-ray photon counts:

$$Y_{t,h} \mid X_t = x_t \sim \mathsf{Poisson}(w \cdot \beta_h \cdot e^{x_t}), \quad h = 1, 2$$

- Smooth transitions in $\{X_t\}$ capture variability in flaring activity as manifested in $\{\boldsymbol{Y}_t\}$

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Likelihood Functions for HMMs

- The vector of parameters $\boldsymbol{\theta}$ in an HMM include those associated with the initial pdf/pmf, the transition pdf/pmf, and the state-dependent distributions
- Suppose we observe data $\boldsymbol{y}_{1:T}$ arising from an HMM
- When $\mathcal{X} = \{1, \ldots, K\}$, the likelihood is a sum over all possible state paths:

$$L(\boldsymbol{\theta} \mid \boldsymbol{y}_{1:T}) = \sum_{x_1=1}^{K} \cdots \sum_{x_T=1}^{K} \delta_{x_1} \cdot f_{x_1}(y_1) \prod_{t=2}^{T} \gamma_{x_{t-1},x_t}^{(t)} \cdot f_{x_t}(y_t)$$

- When $\mathcal{X} = \mathbb{R}^d$, the sums are replaced by integrals:

$$L(\boldsymbol{\theta} \mid \boldsymbol{y}_{1:T}) = \int \cdots \int \delta(x_1) \cdot f_{x_1}(y_1) \prod_{t=2}^{T} \gamma^{(t)}(x_{t-1}, x_t) \cdot f_{x_t}(y_t) \, \mathrm{d}x_{1:T}$$

Fitting HMMs via Likelihood Maximization

- Once an HMM has been specified, it can be fit by maximizing the likelihood:

$$\hat{\boldsymbol{\theta}} = \underset{\boldsymbol{\theta}}{\text{argmax}}\, L(\boldsymbol{\theta} \mid \boldsymbol{y}_{1:T})$$

- For discrete-state HMMs, the likelihood can be computed efficiently via the **forward algorithm** in $O(TK^2)$ time
- For continuous-space HMMs, the likelihood must be approximated numerically (e.g., via state-space discretization [Zimmerman et al., 2024] or particle methods)
- In practice, we optimize the likelihood using numerical methods (e.g., L-BFGS)
  - Transformations ensure parameters stay within valid domains (e.g., $\log$ or $\tanh^{-1}$)

Model Assessment: Pseudo-Residuals

- To assess how well the fitted HMM explains observed *univariate* data, we use **pseudo-residuals**

- These are constructed from the one-step-ahead forecast distribution:

$$r_t = \Phi^{-1}(\mathbb{P}(Y_t \leq y \mid Y_{1:t-1})), \quad t = 2, 3, \ldots, T$$

  The cdf above can either be computed exactly or estimated, depending on the type of HMM

- Under a well-specified model, $r_2, \ldots, r_T$ should be approximately $\mathcal{N}(0, 1)$
  - Deviations reveal distributional misfit or unmodeled serial dependence

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State Decoding: Inferring the Latent Process

- Once we've fit the HMM using an estimator $\hat{\boldsymbol{\theta}}$, we can recover information about the hidden states $\{X_t\}$ using one of two common approaches:
- For discrete-space HMMs: **local decoding**

$$\hat{X}_t = \operatorname*{argmax}_{x \in \mathcal{X}} \mathbb{P}_{\hat{\boldsymbol{\theta}}}(X_t = x \mid \boldsymbol{Y}_{1:T} = \boldsymbol{y}_{1:T}), \quad t = 1, \ldots, T$$

  or **global decoding**

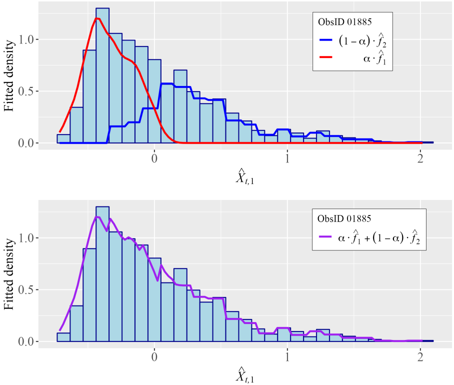
$$\widehat{X}_{1:T} = \operatorname*{argmax}_{x_{1:T} \in \mathcal{X}^T} \mathbb{P}_{\hat{\boldsymbol{\theta}}}(X_{1:T} = x_{1:T} \mid \boldsymbol{Y}_{1:T} = \boldsymbol{y}_{1:T})$$

  These both require the filtered state probabilities $\mathbb{P}_{\hat{\boldsymbol{\theta}}}(X_t = x \mid \boldsymbol{Y}_{1:t} = \boldsymbol{y}_{1:t})$, which can be computed efficiently using the forward algorithm
- For continuous-space HMMs: **posterior expectation**

$$\hat{X}_t = \mathbb{E}_{\hat{\boldsymbol{\theta}}}[X_t \mid \boldsymbol{Y}_{1:T} = \boldsymbol{y}_{1:T}], \quad t = 1, \ldots, T$$

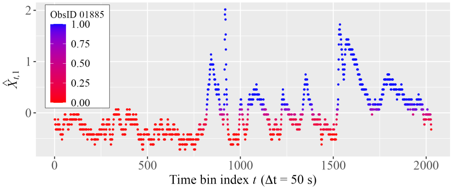
Back to EV Lac

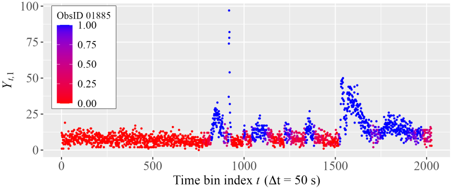
- In the EV Lac model, we compute the smoothed posterior means $\{\hat{X}_t\}$ to estimate the underlying flare intensity at each time point
- We then fit a 2-component mixture model to the distribution of $\{\hat{X}_t\}$:



EV Lac: Flaring and Quiescence

- The fitted mixture model above allows us to estimate the "probability" of flaring for each observation:





Forecasting States Ahead in Time

- Consider the discrete-space HMM and suppose we've made state predictions by computing the filtered state probabilities $\mathbb{P}_{\hat{\boldsymbol{\theta}}}(X_t = x \mid \boldsymbol{Y}_{1:T} = \boldsymbol{y}_{1:T})$

- We can forecast future states conditional on the observed data $\boldsymbol{Y}_{1:T}$ practically for free:

$$\hat{X}_{T+t} = \operatorname*{argmax}_{x \in \mathcal{X}} \sum_{k=1}^{K} \mathbb{P}_{\hat{\boldsymbol{\theta}}}(X_t = k \mid \boldsymbol{Y}_{1:T} = \boldsymbol{y}_{1:T}) \cdot \left[\hat{\boldsymbol{\Gamma}}^t\right]_{k,x}, \quad t = 1, 2, \ldots$$

  where $\hat{\boldsymbol{\Gamma}}$ is the fitted transition probability matrix

- BUT: as $t \to \infty$, the forecast distribution converges to the stationary distribution (regardless of history)
  - So predictive uncertainty increases with t: farther-out forecasts are more diffuse and less informative

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Covariates in State Dependent Distributions

- The basic HMM may be too simplistic a model for certain applications
- Occasionally, we might want certain parameters in the model to depend on covariates (for example, an animal's sex, weight, age, etc.)
- For example, the state-dependent mean θ_x might depend linearly on some fixed vector $\mathbf{z} \in \mathbb{R}^p$, perhaps through some link function g :

$$g(\theta_x) = g(\mathbb{E}[Y_t \mid X_t = x]) = \boldsymbol{\beta}_x^\top \mathbf{z},$$

  where $\boldsymbol{\beta}_x^\top = (\beta_{x,1}, \ldots, \beta_{x,p})$ is a vector of regression coefficients
- In other words, each state-dependent distribution carries its own generalized linear model

Mixed HMMs

- We may have *multiple* time series — say S of them — available for inference
- When the time series are believed to be iid, they can be pooled together in a straightforward manner
- More realistically, the S time series are not iid, but still arise from HMMs with common features (such as the same underlying set of states $\mathcal{X}$)
- When the time series arise from the same parametric model (but with series-specific parameters), there can be up to $S \cdot \text{length}(\boldsymbol{\theta})$ parameters to estimate, which is cumbersome
- For example, there would be S state-dependent parameters for state j: $\theta_{j,1}, \ldots, \theta_{j,S}$

Random Effects

- Instead, one could regard the $\theta_{j,s}$ as continuous random variables: $\theta_{j,1}, \ldots, \theta_{j,S} \stackrel{iid}{\sim} g_{\boldsymbol{\eta}_j}$
- That is, each $\theta_{j,s}$ is a *random effect* with distribution $g_{\boldsymbol{\eta}_j}$
- Each inclusion of such a random effect in the model reduces the number of parameters to estimate by $S - \text{length}(\boldsymbol{\sigma}_j)$
- The drawback, however, is that the $\theta_{j,s}$ must be integrated out of the likelihood:

$$L(\ldots, \boldsymbol{\eta}_j \mid \boldsymbol{y}_{1:T}) = \int \cdots \int L(\ldots, \theta_{j,1}, \ldots, \theta_{j,S} \mid \boldsymbol{y}_{1:T}) \prod_{s=1}^{S} \left(g_{\boldsymbol{\eta}_j}(\theta_{j,s}) \, \mathrm{d}\theta_{j,s} \right)$$

Discrete Random Effects

- Even for the simplest distributions $g_{\boldsymbol{\eta}_j}$, such integrals are never available in closed form and must be computed numerically (which is difficult in high dimensions)
- Alternatively, one can assume the $\theta_{j,s}$ to be *discrete* random variables on a finite sample space $\mathcal{M}$
- This makes for a simpler likelihood computation:

$$L(\ldots, \boldsymbol{\eta}_j \mid \boldsymbol{y}_{1:T}) = \sum_{s=1}^{S} \sum_{m \in \mathcal{M}} L(\ldots, \theta_{j,1}, \ldots, \theta_{j,S} \mid \boldsymbol{y}_{1:T}) \cdot \mathbb{P}_{\boldsymbol{\eta}_j}(\theta_{j,s} = m)$$

- However, the applicability of such models may be limited
- The same ideas can be extended to dependent random effects, in which two or more parameters in the model follow a joint distribution

Covariates in Transition Probabilities

- Alternatively, we may incorporate covariates into the transition pdf/pmf
- In the discrete-state case, this is typically accomplished by applying a multinomial logistic regression model to each row of the transition matrix:

$$\gamma_{j,x} = \mathbb{P}(X_t = x \mid X_{t-1} = j) = \frac{e^{\boldsymbol{\beta}_{x|j}^\top \mathbf{z}}}{1 + \sum_{k=1}^{K-1} e^{\boldsymbol{\beta}_{k|j}^\top \mathbf{z}}}, \quad x, j \in \mathcal{X}$$

with $\boldsymbol{\beta}_{K|j} = \mathbf{0}$ for all $j \in \mathcal{X}$

More on Covariates

- In either case, the $\boldsymbol{\beta}_x$ and/or $\boldsymbol{\beta}_{x|j}$ are incorporated into the likelihood function and inference proceeds as usual
- We might also want to include covariates $\mathbf{z}_t$ that depend on time (for example, $\mathbf{z}_t$ could include the number of hours an animal has been awake at time t)
- In this case, inference proceeds in a similar fashion; however...
- Including time-varying covariates in the transition probabilities $\gamma_{j,x}$ destroys the assumption of time-homogeneity, so the initial pmf $\delta_x = \mathbb{P}(X_0 = x)$ must also be estimated

Bayesian Inference

- One can also perform Bayesian inference on HMMs
- To do so, one must choose an appropriate prior distribution $\pi(\boldsymbol{\theta})$ for the unknown parameters of the model
- In the discrete-space case, the rows of the transition matrix $\boldsymbol{\Gamma}$ and the initial distribution vector $\boldsymbol{\delta}$ are traditionally assigned Dirichlet priors (which are conjugate to the multinomial distribution)
- Priors for the parameters $\boldsymbol{\theta}_x$ of the state-dependent distributions are chosen on a case-by-case basis

Bayesian Inference

- The posterior distribution

$$\pi(\boldsymbol{\theta} \mid \boldsymbol{y}_{1:T}) \propto \pi(\boldsymbol{\theta}) \cdot L(\boldsymbol{\theta} \mid \boldsymbol{y}_{1:T})$$

  is never available in closed form and is impossible to sample from directly

- Thus, Markov chain Monte Carlo (MCMC) methods are typically required to sample from it

- A popular choice of MCMC method for HMMs is Hamiltonian Monte Carlo (or variants thereof), as implemented in the Stan programming language

- Although written in C++, Stan has an R interface which is accessed through the `rstan` library and a Python interface accessed through the `PyStan` library

Quantifying Uncertainty

- As in all statistical inference, it is always of interest to quantify uncertainty in estimates of unknown parameters
- For frequentist inference, asymptotic normality of the MLE has been proven under mild regularity conditions [Bickel et al., 1998]
- The observed information matrix — which itself is a consistent estimator of the Fisher information — can be approximated numerically, and this yields standard errors and confidence intervals for parameter estimates
- In the Bayesian setup, credible intervals can be obtained from posterior distributions using standard techniques

Agenda

When Are HMMs a Good Choice?

- Use an HMM when...
  - You suspect that observed temporal patterns are driven by an unobserved process with temporal structure
  - Your observed data are conditionally independent, given the hidden state
  - You want to classify, decode, or predict based on latent regimes or behaviors
- In astronomy, HMMs are useful for
  - Identifying flaring vs. quiescent periods in light curves
  - Separating source vs. background states in high-energy data
  - Modeling transitions between different emission regimes
  - [Stanislavsky et al., 2020, Zimmerman et al., 2024, Esquivel et al., 2025]
- They can be applied to counts, images, spectra, or multivariate signals
- They can be flexibly extended (e.g., to hierarchical or switching models)

Further Resources

- Introductory and advanced textbooks:
  - [Zucchini et al., 2016]: accessible, example-driven introduction (R-based)
  - [Cappé et al., 2005]: rigorous treatment, theory-heavy (math/stats focused)

- Software for fitting HMMs:
  - In R:
    - `depmixS4`: Discrete-state HMMs with Gaussian, Poisson, multinomial state-dependent distributions
    - `momentuHMM`: Geared toward animal movement, but widely used in practice
    - `hmmTMB`: Flexible HMMs with random effects
    - `nimble`: For custom Bayesian state-space/HMM models with full MCMC
  - In Python:
    - `hmmlearn`: Standard library for discrete-state HMMs (scikit-learn–like)
    - `pomegranate`: Modular, faster implementation for HMMs and other probabilistic models
    - `tensorflow probability`: For building custom probabilistic models (Bayesian HMMs, etc.)

Thank you!

Download the Slides

You can download this presentation at https://rob-zimmerman.github.io/files/presentations/HMM_Tutorial_IACHEC2025.pdf

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